Calculate the limit.
https://www.linkedin.com/feed/update/urn:li:activity:6452843620734242816
Calculate the limit
$\lim_{n\to\infty} \sqrt[n]{n!}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right).$

Solution by Arkady Alt, San Jose, California, USA.

Let $\alpha_n := \frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}$. Noting that $\lim_{n\to\infty}\sqrt[n]{n} = 1$ we obtain

$\lim_{n\to\infty}\alpha_n = \frac{\pi}{4}\lim_{n\to\infty}\frac{n+1}{n}\cdot\lim_{n\to\infty}\sqrt[n+1]{n+1}\cdot\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}} = \frac{\pi}{4}$. Since $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = e$ and

$\tan\alpha_n - 1 = \frac{\sqrt{2}\sin\left(\alpha_n-\frac{\pi}{4}\right)}{\cos\alpha_n}$ then

$\lim_{n\to\infty} \sqrt[n]{n!}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right) = \frac{1}{e}\lim_{n\to\infty} n(\tan\alpha_n - 1) =$

$\frac{\sqrt{2}}{e}\lim_{n\to\infty}\frac{1}{\cos\alpha_n}\cdot\frac{\sin\left(\alpha_n-\frac{\pi}{4}\right)}{\alpha_n-\frac{\pi}{4}}\cdot n\left(\alpha_n-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{e}\cdot\frac{1}{1/\sqrt{2}}\cdot 1\cdot\lim_{n\to\infty} n\left(\alpha_n-\frac{\pi}{4}\right) =$

$\frac{2}{e}\lim_{n\to\infty} n\left(\alpha_n-\frac{\pi}{4}\right) = \frac{2}{e}\cdot\frac{\pi}{4}\lim_{n\to\infty} n\left(\frac{(n+1)\sqrt[n+1]{n+1}}{n\sqrt[n]{n}}-1\right) =$

$\frac{\pi}{2e}\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{n+1} - n\sqrt[n]{n}\right).$

Thus remains to find* $\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{n+1} - n\sqrt[n]{n}\right)$

Let $b_n = (n+1)\sqrt[n+1]{n+1}, a_n = n\sqrt[n]{n}$. Then $\lim_{n\to\infty}\frac{b_n}{a_n} = \lim_{n\to\infty}\frac{(n+1)\sqrt[n+1]{n+1}}{n\sqrt[n]{n}} = 1,$

$\lim_{n\to\infty}\frac{a_n}{n} = \lim_{n\to\infty}\sqrt[n]{n} = 1, \lim_{n\to\infty}\left(\frac{b_n}{a_n}\right)^n = \lim_{n\to\infty}\frac{(n+1)^{\frac{n(n+2)}{n+1}}}{n^{n+1}} =$

$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n+1}\cdot\frac{1}{\sqrt[n+1]{n+1}}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}\cdot\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{n+1}} = e.$

Hence, $\lim_{n\to\infty}\left((n+1)\sqrt[n+1]{n+1} - n\sqrt[n]{n}\right) = \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty}\frac{a_n}{n}\cdot\lim_{n\to\infty}\ln\left(\frac{b_n}{a_n}\right)^n = 1.$

Thus, $\lim_{n\to\infty} \sqrt[n]{n!}\left(\tan\left(\frac{\pi(n+1)\sqrt[n+1]{n+1}}{4n\sqrt[n]{n}}\right)-1\right) = \frac{\pi}{2e}.$

* This limit is particular case of a limit represented in the Lemma 1. in the article: Arkady Alt.-Limits of Lalescu kind sequences with p-hyperfactorial and superfactorial, Journal of Classical Analysis, 2014.